Int. J. Heat Mass Transfer. Vol. 23, pp. 1286–1288

0017-9310/80/0901-1286 $02.00/0

A TRANSFORMATION OF THE BOUNDARY LAYER EQUATIONS FOR FREE CONVECTION PAST A VERTICAL FLAT PLATE WITH ARBITRARY BLOWING AND WALL TEMPERATURE VARIATIONS

M. VEDHANAYAGAM, R. A. ALTENKIRCH and R. EICHHORN
College of Engineering, University of Kentucky, Lexington, Kentucky 40506, U.S.A.

(*Received* 13 *November* 1979 *and in revised form* 26 *February* 1980)

NOMENCLATURE

f, dimensionless stream function;
F, stream function;
g, gravitational acceleration;
Pr, Prandtl number;
Q, dimensionless heat flux;
T, temperature;
u, x velocity;
v, y velocity;
x, y, axial and normal distance coordinates.

Greek symbols

β, coefficient of thermal expansion;
$\tilde{\beta}$, dimensionless wall temperature parameter;
$\tilde{\gamma}$, dimensionless blowing rate;
ΔT, temperature difference ($= T_w - T_\infty$);
η, dimensionless normal coordinate;
θ, dimensionless temperature;
ν, kinematic viscosity;
ξ, dimensionless streamwise coordinate;
τ, dimensionless shear stress;
ψ, stream function.

Subscripts

w, wall;
∞, ambient.

INTRODUCTION

FREE convection with mass transfer at the wall has been studied by many authors in the past. Eichhorn [1] reported similarity solutions to the boundary layer equations for the case of free convection along a vertical flat plate with blowing or suction at the wall. For the case of constant wall temperature with uniform blowing, where similarity solutions are not possible, Sparrow and Cess [2] provided approximate series solutions, Merkin [3] and Parikh *et al.* [4] used finite difference techniques to solve the nonsimilar boundary layer equations and more recently Minkowycz and Sparrow [5], using the local nonsimilarity method, presented solutions for a wide range of Prandtl numbers. Clarke [6], allowing for variable density, calculated the induced outer flow for similarity blowing at the wall. Kao *et al.* [7] presented solutions for free convection along a vertical plate with arbitrary wall temperature variations in the absence of mass transfer at the wall.

In this note we present a transformation of the laminar boundary layer equations, similar to the one of Kao *et al.*, which allows arbitrary distributions of both wall temperature and blowing. The procedure yields constant boundary conditions, but variable coefficients appear in the differential equations. The transformed equations can be solved using a variety of techniques. Here we present only the local similarity solutions as a general case and as a first approximation to the solution to the full problem since our main purpose here is simply to make known the transformation.

The impetus for our seeking such a transformation stems from our interest in the downward burning of vertical pieces of condensed-phase combustibles. For flames propagating over solid combustibles, the surface temperature varies along the direction of spread. Solid vaporization occurs by a kinetic law of the Arrhenius type, rather than by maintenance of evaporative equilibrium that occurs for liquid fuels, so that the blowing rate is in part determined by the surface temperature distribution.

ANALYSIS AND RESULTS

The problem is governed by the usual constant property boundary layer equations with the Boussinesq body-force term in the x-momentum equation. We take u and v to be the x and y velocity components and T as the temperature. At the surface of the plate, where $y = 0$, u is zero, v is $v_w(x)$ and T is $T_w(x)$. For y large, u is zero and the temperature is equal to the constant ambient value, T_∞, which we take to be less than $T_w(x)$ so that the fluid flow is in the positive x direction.

Continuity is satisfied by the introduction of a stream function, ψ, and we introduce the following transformations to incorporate the boundary conditions in the boundary layer equations:

$$\left.\begin{aligned}
\psi &= -\int_0^x v_w(x)\,\mathrm{d}x + F(x,y),\\
\xi &= \beta\left(\frac{g}{\nu^2}\right)^{1/3}\int_0^x (T_w(\chi) - T_\infty)\,\mathrm{d}x,\\
\eta &= \beta^{1/2}\left(\frac{g}{4^{3/4}\nu^2}\right)^{1/3}(T_w(x) - T_\infty)^{1/2}\,y\,\xi^{-1/4},\\
F &= \nu(4\xi)^{3/4} f(\xi,\eta)\beta^{-1/2}(T_w(x) - T_\infty)^{-1/2},\\
\theta(\xi,\eta) &= (T - T_\infty)/(T_w(x) - T_\infty).
\end{aligned}\right\} \quad (1)$$

These transformations reduce the boundary layer equations to:

$$f''' + (3 - 2\tilde{\beta})ff'' - \tilde{\gamma}f'' - 2(f')^2 + \theta = 4\xi\left[f'\frac{\partial f'}{\partial \xi} - f''\frac{\partial f}{\partial \xi}\right],$$

$$\frac{\theta''}{Pr} + (3 - 2\tilde{\beta})f\theta' - \tilde{\gamma}\theta' - 4\tilde{\beta}f'\theta = 4\xi\left[f'\frac{\partial \theta}{\partial \xi} - \theta'\frac{\partial f}{\partial \xi}\right],$$

where

$$\tilde{\beta}(\xi) = \frac{\xi\left(\dfrac{\nu^2}{g}\right)^{1/3}}{\beta(T_w(x) - T_\infty)^2}\,\frac{\mathrm{d}T_w(x)}{\mathrm{d}x},$$

$$\tilde{\gamma}(\xi) = v_w(x)\left[\frac{4\xi\left(\dfrac{\nu^2}{g}\right)^{1/3}}{g\beta^2\nu^2(T_w(x) - T_\infty)^2}\right]^{1/4} \quad (2)$$

Table 1. Dimensionless skin friction and heat transfer for $Pr = 0.7$

$\tilde{\beta}$ \ $\tilde{\gamma}$		−1.0	−0.8	−0.6	−0.4	−0.2	0.0	0.2	0.4	0.6	0.8	1.0
−1.6	$f''(0) =$	0.674	0.693	0.709	0.723	0.734	0.743	0.750	0.755	0.757	0.758	0.757
	$-\theta'(0) =$	0.684	0.541	0.398	0.253	0.108	−0.039	−0.186	−0.335	−0.486	−0.640	−0.780
−1.4		0.675	0.692	0.707	0.719	0.729	0.735	0.739	0.741	0.740	0.736	0.730
		0.716	0.579	0.442	0.307	0.172	0.038	−0.095	−0.226	−0.356	−0.484	−0.610
−1.2		0.675	0.692	0.705	0.716	0.723	0.728	0.729	0.727	0.722	0.714	0.704
		0.747	0.616	0.487	0.360	0.235	0.112	−0.007	−0.123	−0.235	−0.342	−0.444
−1.0		0.676	0.691	0.703	0.712	0.718	0.720	0.718	0.713	0.705	0.694	0.680
		0.779	0.654	0.531	0.412	0.296	0.184	0.076	−0.027	−0.124	−0.214	−0.297
−0.8		0.676	0.690	0.701	0.709	0.712	0.712	0.708	0.700	0.689	0.674	0.657
		0.811	0.691	0.575	0.463	0.356	0.253	0.155	0.064	−0.021	−0.099	−0.168
−0.6		0.676	0.690	0.699	0.705	0.706	0.704	0.697	0.687	0.673	0.656	0.635
		0.842	0.729	0.619	0.514	0.414	0.319	0.230	0.148	0.073	0.005	−0.054
−0.4		0.677	0.689	0.697	0.701	0.700	0.695	0.686	0.674	0.657	0.638	0.615
		0.874	0.766	0.662	0.563	0.470	0.382	0.301	0.226	0.158	0.098	0.045
−0.2		0.677	0.688	0.695	0.697	0.694	0.687	0.676	0.661	0.642	0.621	0.597
		0.907	0.803	0.705	0.612	0.524	0.442	0.367	0.298	0.235	0.180	0.131
0.0		0.678	0.687	0.692	0.693	0.688	0.679	0.665	0.648	0.628	0.605	0.580
		0.939	0.840	0.747	0.659	0.576	0.499	0.429	0.364	0.305	0.252	0.205
0.2		0.678	0.687	0.690	0.688	0.682	0.671	0.655	0.636	0.614	0.590	0.564
		0.971	0.877	0.788	0.705	0.626	0.553	0.486	0.424	0.367	0.315	0.269
0.4		0.679	0.686	0.688	0.684	0.675	0.662	0.645	0.624	0.601	0.576	0.550
		1.003	0.914	0.829	0.749	0.674	0.604	0.539	0.478	0.422	0.370	0.324
0.6		0.679	0.685	0.685	0.679	0.669	0.654	0.634	0.613	0.589	0.564	0.537
		1.036	0.950	0.869	0.792	0.719	0.651	0.587	0.527	0.470	0.418	0.370
0.8		0.680	0.684	0.682	0.675	0.662	0.645	0.625	0.602	0.578	0.552	0.525
		1.068	0.986	0.907	0.833	0.762	0.695	0.631	0.570	0.513	0.460	0.411
1.0		0.680	0.683	0.679	0.669	0.655	0.637	0.615	0.592	0.567	0.541	0.515
		1.101	1.021	0.945	0.872	0.802	0.734	0.670	0.609	0.551	0.497	0.446

The boundary conditions become:

$$f'(\xi,0) = f(\xi,0) = f'(\xi,\infty) = \theta(\xi,0) = 0, \quad \theta(\xi,0) = 1. \tag{3}$$

In equations (1), (2) and (3) the prime notation denotes differentiation with respect to η, β is the volume coefficient of expansion, g is the acceleration of gravity, v is the kinematic viscosity and Pr is the Prandtl number. In equation (2) x is to be interpreted as the function of ξ defined by the integral relationship written for ξ in equation (1). The differential equations for f and θ reduce to those of Kao *et al.* for $\tilde{\gamma}$ equal to zero and to those of Eichhorn under similarity conditions. Data for a specific problem of interest are contained in the parameters $\tilde{\beta}$ and $\tilde{\gamma}$ rather than in the boundary conditions.

The local similarity solution, in the form of wall shear stress and heat transfer rate, to equations (2) and (3) for a Prandtl number of 0.7 and various values of $\tilde{\beta}$ and $\tilde{\gamma}$ is given in Table 1. In this approximation, valid at small ξ, the right-hand side of the differential equations for f and θ is dropped, and the resulting ordinary differential equations, along with the boundary conditions, are solved with $\tilde{\beta}$, $\tilde{\gamma}$ and Pr as parameters. To solve the equations we used the shooting technique described by Nachtsheim and Swigert [8].

As an example of how results such as those in Table 1 might be used, consider the case of an isothermal wall with uniform blowing, the problem previously considered by Merkin [3]. From equation (2) we get

$$\tilde{\beta} = 0, \quad \tilde{\gamma} = v_w \left[\frac{4x}{g\beta v^2 \Delta T}\right]^{1/4}, \tag{4}$$

where $\Delta T = (T_w - T_\infty)$, a constant. The local heat transfer rate and skin friction can now be expressed in terms of the dimensionless parameters:

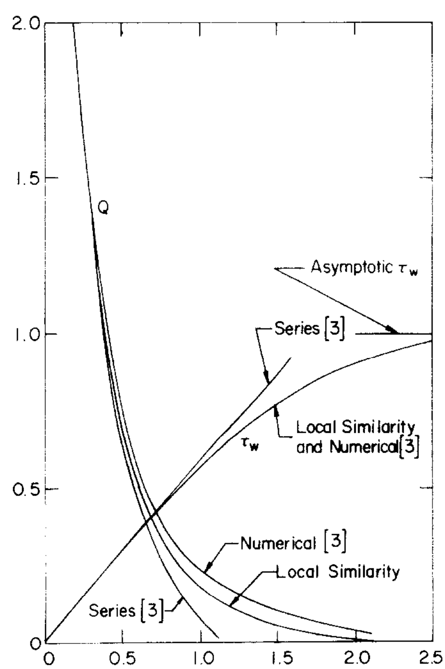

FIG. 1. Free convection on an isothermal plate with uniform blowing.

heat transfer rate

$$Q = -\frac{v}{v_w \Delta T}\frac{\partial T}{\partial y}\bigg|_{y=0} = -\frac{1}{\tilde{\gamma}}\theta'(0),$$

skin friction (5)

$$\tau_w = \frac{v_w}{g\beta\Delta T}\frac{\partial u}{\partial y}\bigg|_{y=0} = \tilde{\gamma} f''(0).$$

These parameters and the notation are the ones used by Merkin [3] in his numerical solution, and our $\tilde{\gamma}$ corresponds to his dimensionless streamwise coordinate. Values of $\theta'(0)$ and $f''(0)$ were recomputed for $Pr = 1$, and the results are plotted, along with those from [3], in Fig. 1. The local similarity solution approximates the numerical solution quite well and is somewhat better than the series solution.

It is interesting to note in Table 1 that at high negative values of $\tilde{\beta}$, combined with high values of $\tilde{\gamma}$, the surface heat flux is negative, i.e. to the plate rather than from it. Physically this occurs when the surface temperature falls so rapidly in the flow direction that the fluid convected along the wall is at a higher temperature than the wall at that x-location. The extent to which such a local similarity prediction is correct depends on the smallness of ξ and the applicability of the boundary layer equations themselves.

CONCLUSION

The simple transformations introduced by Kao *et al.* [7], which lead to constant boundary conditions, can be extended to include the effect of variable blowing rate at the wall.

Acknowledgement — We would like to thank the National Aeronautics and Space Administration for supplying funds for this work under NASA-Lewis Grant NSG-3114 with T. H. Cochran as project manager.

REFERENCES

1. R. Eichhorn, The effect of mass transfer on free convection, *J. Heat Transfer* **82C**, 260–263 (1960).
2. E. M. Sparrow and R. D. Cess, Free convection with blowing or suction, *J. Heat Transfer* **81C**, 387–389 (1960).
3. J. H. Merkin, Free convection with blowing and suction, *Int. J. Heat Mass Transfer* **15**, 989–999 (1972).
4. P. G. Parikh, R. J. Moffat, W. M. Kays and D. Bershader, Free convection over a vertical porous plate with transpiration, *Int. J. Heat Mass Transfer* **17**, 1465–1474 (1974).
5. W. J. Minkowycz and E. M. Sparrow, Numerical solution scheme for local nonsimilarity boundary-layer analysis, *Numerical Heat Transfer* **1**, 69–85 (1978).
6. J. F. Clarke, Transpiration and natural convection — vertical plate problem, *J. Fluid Mech.* **57**, part 1, 46–61 (1973).
7. T. Kao, G. A. Domoto and H. G. Elrod, Free convection along a non-isothermal vertical flat plate, *J. Heat Transfer* **99C**, 72–78 (1977).
8. P. R. Nachtsheim and P. Swigert, Satisfaction of asymptotic boundary conditions in numerical solution of systems of non-linear equations of boundary layer type, NASA TN-D3004 (1965).